Armed Services Technical Information Agency

AD 38698

UNCLASSIFIED

A THEORY OF THE SOLAR MAGNETIC FIELD

BY

EUGENE N. PARKER

TECHNICAL REPORT NO. 6

August 1, 1954

DEPARTMENT OF PHYSICS

UNIVERSITY OF UTAH

SALT LAKE CITY

A Theory of the Solar Magnetic Field

Eugene N. Parker
Department of Physics
University of Utah

Abstract

Using the basic principles of operation of the ter-
restrial hydromagnetic dynamo, it is shown that the solar convec-
tive zone will generate traveling dynamo waves consisting of
toroidal and poloidal magnetic fields. The waves migrate from
the poles toward the equator near the top of the convective zone
and in the opposite direction near the bottom. The thinness
of the convective zone is responsible for a migratory dynamo
in the sun rather than a stationary dynamo as in the Earth. An
investigation of the dynamics of hydromagnetic dynamos indicates
that the poloidal field will be strongest above middle latitudes
and the toroidal field below, in agreement with the observed
nonuniform rotation and the appearance of sunspots in low
latitudes.

1. Introduction

The migration toward the equator and the reversal of polarity of sunspots each eleven year half-cycle are probably the most conspicuous aspects of the overall solar magnetic activity. The emission of excessive radio noise and charged particles, prominence and flare activity, and coronal streamers, as well as direct observation of the magnetic field over the disk of the sun, (Babcock, 1953) indicate that the solar magnetic field is complex - certainly not just a dipole - and has a nearly periodic time dependence.

The Earth's field, which is predominantly a dipole, with its moderate fluctuations in space and time, is satisfactorily accounted for by a stationary hydromagnetic dynamo model based on the convective motions in the conducting fluid core and the Coriolis forces resulting from the Earth's diurnal rotation (Elsasser, 1946/7; Parker, 1954). The two necessary dynamo conditions, viz. rotation of, and convection within, a large electrically conducting fluid body, are satisfied by the sun. This suggests that the solar magnetic field is the result of a hydromagnetic dynamo; it will be our purpose here to construct a crude, appropriate model.

We have at our disposal a convective layer near the surface of the sun with a thickness of the order of 10^5 km. The convection results in the first place in nonuniform rotation with the outer parts of the convective layer lagging behind the inner. These fluid motions also seem to produce a poloidal magnetic field

which has only a small dipole component, which is of opposite sign in the northern and southern hemispheres, which undergoes continual migration toward the equator from both poles, and which reverses sign at any given point on the sun every eleven years. This suggests "waves" of magnetic flux originating near the poles and being propagated regeneratively toward the equator where they are dissipated; the period of the waves would be 22 years.

2. Qualitative Discussion

Before developing an elaborate analysis of a dynamo model
satisfying these requirements, let us develop the model in a
qualitative way in order to understand physically the hydromagnetic
processes we might expect to occur. (The calculations of the next
section will be of interest to the person interested especially
in dynamos.) Figure 1 shows a slab of fluid representing a
piece of the convective shell of the sun. At time t_o we take
the section to be a rectangular parallelepiped, neglecting the
curvature, as indicated by the broken lines. At some later time
the nonuniform rotation will have sheared the section into the
form shown by the solid lines, and a loop of flux initially in
a meridional plane will be carried by the fluid into the nearly
horizontal position represented by the unbroken elliptical band.
This is just the familiar process of the nonuniform rotation
generating a toroidal field from the poloidal field. The net
poloidal field is unchanged by the process.

Consider next a meridional section through the convective
zone illustrated in figure 2(a). Two initial magnetic loops in
the meridional plane are represented by the heavy closed lines.
We regard the meridional section illustrated in figure 2 as the
left hand end of the slab in figure 1, so that the nonuniform
rotation displaces the top side of the meridional section in
figure 2 out of the paper relative to the bottom. The toroidal
field will be out of the paper near the center and into the paper
at the right and left hand ends, as indicated by the plus and minus
signs respectively. The convective motions interacting with the

toroidal fields will produce magnetic loops with nonvanishing
projections on the meridional planes (Parker, 1954). The sense
of these loops depends upon the sense of the toroidal field. Let
us assume that when the toroidal field is directed out of the paper
the individual loops are in a counterclockwise direction, when
the toroidal field is into the paper, the loops will be in a clock-
wise direction. Many such loops will be produced, and it can be
shown that they will coalesce into large loops, which are indi-
cated by the light lines in figure 2(a). If now we add the newly
formed large loops to the initial loops, we see from the sense of
rotation that the initial loops will be reinforced in the vicinity
of B and D and degenerated at A and C. Figure 2(b) shows the
result of this superposition, resulting in the initial loops being
displaced to the right. Thus, we have a dynamo which not only may
have regenerative feedback between the poloidal and toroidal fields,
but which also produces a migration of the field. The direction
of the migration depends, of course, on the sense of the loops
and of the toroidal field.

3. The Migratory Dynamo Equations

Consider the space determined by the rectangular coordinates (x, y, z) and filled with a fluid with magnetic viscosity

$$\nu = \frac{1}{\mu \sigma}$$

We shall assume that the fluid has several motions: There is a motion $\underline{e}_y \, v(x,z)$, where $\underline{e}_x$, $\underline{e}_y$, and $\underline{e}_z$ are unit vectors in the x, y and z directions respectively. $\partial v(x,z)/\partial z$ will represent the shearing of the nonuniform rotation found in stellar convective zones. Besides $v(x,z)$ we will assume that the fluid is convecting in columns parallel to the z -axis. The columns are rotating about their axes as they would in an actual star because of Coriolis forces. To compare this idealized rectangular model with the actual physical situation in a star, we will regard some cell, say $0 < x < \alpha$, $0 < z < \beta$ as a piece of the convective shell in which the curvature has been neglected; we will think of the positive x-axis as pointing to the south, the y-axis to the east, and the z-axis upward.

We shall represent the toroidal field by $\underline{B}$, where

$$\underline{B} = \underline{e}_y \, B(r,t) \quad , \quad \underline{r} = \underline{e}_x \, x + \underline{e}_z \, z \tag{1}$$

The poloidal field is more easily represented by the vector potential

$$\underline{A} = \underline{e}_y \, A(r,t) \tag{2}$$

than by the magnetic field itself, which has both z and x components. We shall take

$$\frac{\partial B}{\partial y} = \frac{\partial A}{\partial y} = 0 \tag{3}$$

The interacting with the toroidal field $\mathbf{B}$ of a rotating convective column moving in the z direction generates a magnetic loop with a nonvanishing projection on the xz-plane. The details of the loop generation may be found in an earlier paper (1954). Briefly, what occurs is that a rotating or cyclonic convective column rises up through the toroidal field poking the field ahead of it to make a bump. The rotation of the convective column then twists the bump through 90° giving magnetic loops in the meridional plane. The loops coalesce to form the poloidal field. Since we represent the poloidal field by its vector potential, each loop will appear as a hump in the function $A(x, t)$. If we assume that the rate of generation of $A(x, t)$ at any given point is proportional to the toroidal field from which it is generated, we obtain

$$\frac{\partial A}{\partial t} - \nu \nabla^2 A = \Gamma(x) \, B \tag{4}$$

where $\Gamma(x)$ is some function characteristic of the convective motions.

The toroidal field is generated through the interaction of $v(x,z)$ with the poloidal field. Hence

$$\frac{\partial}{\partial t} \mathbf{B} - \nu \nabla^2 \mathbf{B} = \nabla \times \left[\mathbf{v} \times (\nabla \times \mathbf{A}) \right] \tag{5}$$

But since $\mathbf{v}$ and $\mathbf{A}$ have x and z components only

$$\nabla \times \left[\mathbf{v} \times (\nabla \times \mathbf{A}) \right] = \mathbf{e}_y \left(\frac{\partial v}{\partial z} \frac{\partial A}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial A}{\partial z} \right) = \mathbf{e}_y (\nabla v) \times (\nabla A) \tag{6}$$

Hence

$$\frac{\partial B}{\partial t} - \nu \nabla^2 B = (\nabla v) \times (\nabla A) \tag{7}$$

(4) and (7) represent the complete dynamo equations.[*]) To
effect their solution we define $a(\underline{k},t)$, $b(\underline{k},t)$, $h(\underline{k})$ and $\gamma(\underline{k})$ as the
Fourier transforms of $A(\underline{r},t)$, $B(\underline{r},t)$, $\nabla v(\underline{r})$, and $\Gamma(\underline{r})$ respectively,
so that

$$\left.\begin{array}{l} A(\underline{r},t) = \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \, exp(i\underline{k}\cdot\underline{r})\, a(\underline{k},t) \\[3mm] a(\underline{k},t) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dz \, exp(-i\underline{k}\cdot\underline{r})\, A(\underline{r},t) \end{array}\right\} \quad (8)$$

etc. where

$$\underline{k} = \underline{e}_x k_1 + \underline{e}_z k_2 \qquad (9)$$

We note that

$$\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dz\, exp(-i\underline{k}\cdot\underline{r})\Gamma(\underline{r})B(\underline{r}) = \int_{-\infty}^{+\infty}dk_1'\int_{-\infty}^{+\infty}dk_2'\,\gamma(\underline{k}')b(\underline{k}-\underline{k}')$$

and

$$\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dz\, exp(-i\underline{k}\cdot\underline{r})(\nabla v)\times(\nabla A) = \int_{-\infty}^{+\infty}dk_1'\int_{-\infty}^{+\infty}dk_2'\, b(\underline{k}-\underline{k}')\times i\underline{k}'a(\underline{k}')$$

Operating on (4) and (7) with

$$\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dz\, exp(-i\underline{k}\cdot\underline{r})$$

we obtain the two equations

$$\frac{\partial}{\partial t}a(\underline{k},t) + \nu k^2 a(\underline{k},t) = \int_{-\infty}^{+\infty}dk_1'\int_{-\infty}^{+\infty}dk_2'\,\gamma(\underline{k}')b(\underline{k}-\underline{k}',t) \quad (10)$$

$$\frac{\partial}{\partial t}b(\underline{k},t) + \nu k^2 b(\underline{k},t) = \int_{-\infty}^{+\infty}dk_1'\int_{-\infty}^{+\infty}dk_2'\, h(\underline{k}-\underline{k}')\times i\underline{k}'a(\underline{k}',t) \quad (11)$$

[*]) The integral equation (13) was first derived by the author
from a direct consideration of the toroidal and poloidal fields
and their interaction with the fluid velocity. After reviewing
this rather lengthy derivation, Professor W. M. Elsasser pointed
out that the interaction could be condensed to the two differential
equations (4) and (7), thereby greatly simplifying the development.

The general solution of (10) is

$$a(\underline{k},t) = \int_{-\infty}^{+\infty} dk_1' \int_{-\infty}^{+\infty} dk_2' \; \gamma(\underline{k}') \int_{-\infty}^{t} dt' \exp\left[\nu k^2 (t'-t)\right] b(\underline{k}-\underline{k}',t) \quad (12)$$

Putting (12) into (11) we obtain finally

$$\frac{\partial}{\partial t} b(\underline{k},t) + \nu k^2 b(\underline{k},t) = \quad (13)$$

$$\int_{-\infty}^{+\infty} dk_1' \int_{-\infty}^{+\infty} dk_2' \; b(\underline{k}-\underline{k}') \times i\,\underline{k}' \int_{-\infty}^{+\infty} dk_1'' \int_{-\infty}^{+\infty} dk_2'' \; \gamma(\underline{k}'') \int_{-\infty}^{t} dt' \exp\left[\nu k'^2 (t'-t)\right] b(\underline{k}'-\underline{k}'',t')$$

4. Traveling Dynamo Waves

Let us consider the solution of the dynamo equation, (13). To establish that traveling dynamo waves exist, consider the case where the convective motions and shearing of the nonuniform rotation are independent of position. Then

$$\nabla v = e_z H \; , \quad \Gamma(r) = \Gamma \tag{14}$$

where H and Γ are constants. Then

$$h(k) = e_z H \; \delta(k), \quad \gamma(k) = \Gamma \; \delta(k) \tag{15}$$

where $\delta(k)$ is a Dirac delta function. (13) reduces to

$$\frac{\partial}{\partial t} b(k,t) = -\nu k^2 b(k,t) + i \, H \Gamma \, k_1 \int_{-\infty}^{t} dt' \exp\left[\nu k^2(t'-t)\right] b(k,t') \tag{16}$$

It is readily shown that the solution of (16) may be written

$$b(k,t) = p(k) \exp\left\{ \left[\left(\frac{k_1 |H\Gamma|}{2}\right)^{1/2} - \nu k^2 \right] t \pm i \left(\frac{k_1 |H\Gamma|}{2}\right)^{1/2} t \right\}$$

$$+ q(k) \exp\left\{ -\left[\left(\frac{k_1 |H\Gamma|}{2}\right)^{1/2} + \nu k^2 \right] t \mp i \left(\frac{k_1 |H\Gamma|}{2}\right)^{1/2} t \right\} \tag{17}$$

The $\pm$ has the same sign as k_1 if $H\Gamma > 0$ and the opposite sign if $H\Gamma < 0$. (17) represents a plane wave propagating with velocity $\left[H\Gamma /(2 k_1) \right]^{1/2}$ in the k direction. The second term represents degeneration of the existing field and vanishes for large values of t. Thus, for large values of t, (17) reduces to

$$b(k,t) = p(k) \exp\left\{ \left[\left(\frac{k_1 |H\Gamma|}{2}\right)^{1/2} - \nu k^2 \right] t \pm i \left(\frac{k_1 |H\Gamma|}{2}\right)^{1/2} t \right\} \tag{18}$$

This represents a plane wave traveling in the negative k direction

if $H\Gamma > 0$ and in the positive k direction if $H\Gamma < 0$. The amplitude of the mode with wave number k is constant if

$$|H\Gamma| = \frac{2\nu^2 k^4}{k_1} \tag{19.}$$

the amplitude grows exponential with time if $|H\Gamma|$ is greater than $2\nu^2 k^4/k_1$ and decays if it is less.

From (13) we see in general that the modes k and k' are coupled through the quantities $h(k-k')\gamma(k'-k'')$. Thus the modes will not be independent except in the special case of (14). Coupling between modes increases the difficulty of solving (13) and the general case is discussed only briefly in the appendix, where two special cases are worked out.

5.   The Kinematics of the Solar Dynamo

Consider the cyclonic motion of a rising convective column in the northern hemisphere of the solar convective zone. The Coriolis forces resulting from the influx of matter at the bottom end of the column give rise to a counterclockwise rotation about a vertical axis as viewed from outside the convective zone. Unless dissipated by viscosity the rotation of the fluid will continue until it is brought to a halt by the Coriolis forces of opposite sense due to the efflux at the top of the column. This was the situation assumed in the core of the Earth, illustrated in figure 3(a). The convective motions in the sun, however, are much more rapid than in the core of the Earth, 1 m/sec as compared to 0.1 mm/sec, and of much larger scale, with the result that the solar Reynolds numbers are large. Thus, the motions in the solar convective zone constitute well developed turbulence in contrast to the irregular convective, though not fully turbulent, fluid motions in the core of the Earth. The cyclonic motions in the sun have to contend with eddy viscosity besides the relatively neg-ligible molecular viscosity. Now an eddy in a fully developed field of turbulence has a lifetime of the order of its diameter divided by its velocity, i.e. an eddy lasts about $1/\pi$ of one "revolution". Thus, we expect the cyclone resulting from the in-flux at the base of a rising convective column in the sun to be dissipated sometime before it has made one complete revolution. The efflux at the upper end of the column will then give rise to a new rotational motion, this time an anticyclone. This means

that in the northern hemisphere of the solar convective zone we expect a counterclockwise rotation to be associated with an influx of matter to a convective column and a clockwise rotation with an efflux, as viewed from outside the sun. Vice versa in the southern hemisphere. The rotation of the fluid in a rising convective column in the northern hemisphere is illustrated in figure 3(b)

To state this matter a little differently, we note that an influx is associated with the lower half of rising columns, and an efflux with the upper half. Hence, in the outer half of the solar convective zone, rising and sinking columns are associated with clockwise and counterclockwise rotations respectively, and vice versa in the inner half. Figure 4 illustrates the loops that will be produced in the inner and outer halves of the convective zone assuming a toroidal field running from west to east. We note that in the outer half, the loops due to both sinking and rising columns are in a counterclockwise direction and in the inner half clockwise, as viewed from the east.

Consider the hydromagnetic dynamo that will result from the convective motions outlined above. As mentioned earlier the turbulence within the convective zone gives a large magnetic viscosity or diffusivity compared to the magnetic viscosity of the quiescent matter above and below the zone. Thus, the field is confined to the convective zone, resulting in traveling dynamo waves rather than a stationary field such as one finds in the Earth where the field diffuses out of the core. From qualitative considerations such as illustrated in figure 2 or from the

analysis in sections 3 and 4,[*] we find that the waves will travel toward the equator in the outer half and toward the poles in the inner half of the solar convective zone. The velocity of propagation must be of the order of 2 m/sec in order to give the observed 22-year magnetic period.

A wave in the inner half will decay upon reaching the pole due to the lack of nonuniform rotation. It will introduce only a small perturbation into the outer half, which will be amplified as it propagates back to the equator. As the wave nears the equator it will decay because of the decrease of the cyclonic and anticyclonic motions associated with the radial convective motions. The wave again serves only to introduce a small perturbation down into the lower half of the zone, which is then amplified as it propagates toward the pole.

Initiating the wave at the polar end of the outer half or the equatorial end of the inner half of the convective zone is obviously an inefficient process. Hence, considerable amplification is required en route. Thus we conclude from (18) that

$$\left| \frac{k_1 H \Gamma}{2} \right| \gg \nu^2 k^4 \tag{20}$$

Neglecting νk^2 in (18) the amplification is $\exp\left[t \left| k_1 H \Gamma / 2 \right|^{1/2} \right]$, and the velocity of propagation is $\left| H \Gamma / (2 k_1) \right|^{1/2}$. Thus, if t is the time required to propagate the wavelength $2\pi / k_1$, we have

$$t \left| \frac{H \Gamma}{2 k_1} \right|^{1/2} = \left| \frac{2 \pi}{k_1} \right|$$

---

[*] In figure 3, for instance, $T(r) > 0, \gamma > 0, B > 0, H < 0$ in the upper half of the convective zone. Thus $H \Gamma < 0$ and the wave propagates in the positive x or southerly direction.

or

$$t \left| \frac{k_1 H \Gamma}{2} \right|^{1/2} = 2\pi \tag{21}$$

The amplification is $\exp(2\pi)$. The sunspots and other solar magnetic activity indicate that the 10^6 km distance from the polar to the equatorial regions is about one wavelength, $2\pi/k_1$. Thus, the amplification is of the order of $\exp(2\pi)$ or 500 in each half of the convective zone. Assuming that this amplification is sufficient to make up for the losses, we see that the initial fields, however small, will be built up after a number of cycles to where they limit the motions resulting from the Coriolis forces and reduce $|k_1 H \Gamma/2|$ relative to $\nu^2 k^4$. When this point is reached, the dynamo henceforth operates in a dynamically stable state.

If more were known concerning the dynamical state of the convective zone, it would be fruitful to use turbulence theory to relate $H\Gamma$ and νk^2 to the convective velocities, etc. Suffice it to say that if the depth of the convective zone is 10^5 km, convective velocities of the order of 1 m/sec seem to be ample.

6. The Dynamics of Hydromagnetic Dynamos.

Let us investigate in a semi-quantitative manner the dynamics of hydromagnetic dynamos, both stationary and migratory. Consider a fluid body of magnetic viscosity ν and angular velocity ω. Let the fluid be convecting radially with a mean velocity v. We denote the mean toroidal field by B_t and the mean poloidal field by B_p. Let the scale of the fields be L. Finally, let θ be the angular distance from the north pole.

The force per unit volume exerted on the fluid by a magnetic field $\underline{B}$ is $(\nabla \times \underline{B}) \times \underline{B}/\mu$. Thus, the force exerted by the toroidal field against the nonuniform rotation drawing out the toroidal field from the poloidal field is the φ component of $(\nabla \times \underline{B}) \times \underline{B}/\mu$ and will be represented by $B_t B_p / (\mu L)$. The force exerted by the magnetic field opposing the formation of a loop from the toroidal field will be represented by the same expression, $B_t B_p /(\mu L)$. The Coriolis force available to draw out the toroidal field is $2\rho\omega v \alpha \sin\theta$ and to kink the toroidal field to form loops $2\rho\omega v \beta \cos\theta$, where α and β are dimensionless constants. We see that to draw out the toroidal field we must have

$$2\rho\omega v \alpha \sin\theta > \frac{B_t B_p}{\mu L} \; ;$$

to produce loops

$$2\rho\omega v \beta \cos\theta > \frac{B_t B_p}{\mu L}$$

Were it not for the magnetic field inhibiting the Coriolis motions, the velocity appearing in the generating term $\nabla \times (\underline{v} \times \underline{B})$ would be proportional to the Coriolis forces and we would write

$$\frac{c}{L}\left(2\rho\omega v\alpha\sin\theta\right)B_p \quad \text{and} \quad \frac{b}{L}\left(2\rho\omega v\beta\cos\theta\right)B_t$$

for the rate of generation of the toroidal and poloidal fields respectively; b and c are constants. However, the inhibition of the Coriolis motions by the magnetic field is not negligible, and we shall therefore consider as effective only that fraction of the Coriolis forces which is in excess of the inhibiting magnetic forces. The generating terms of the toroidal and poloidal fields become

$$\frac{c}{L}B_p\left[2\rho\omega v\alpha\sin\theta - \frac{B_t B_p}{\mu L}\right] \quad \text{and} \quad \frac{b}{L}B_t\left[2\rho\omega v\beta\cos\theta - \frac{B_t B_p}{\mu L}\right]$$

The dissipation $\nu\nabla^2 \mathbf{B}$ may be written as $\nu B_t / L^2$ and $\nu B_p / L^2$ for the toroidal and poloidal fields, respectively.

The time-dependent equations for B_t and B_p are

$$\frac{\partial B_t}{\partial t} = -\frac{\nu}{L^2}B_t + \frac{c}{L}B_p\left[2\rho\omega v\alpha\sin\theta - \frac{B_t B_p}{\mu L}\right]$$

$$\frac{\partial B_p}{\partial t} = -\frac{\nu}{L^2}B_p + \frac{b}{L}B_t\left[2\rho\omega v\beta\cos\theta - \frac{B_t B_p}{\mu L}\right]$$

(22)

Consider first the equilibrium values of B_t and B_p, which we shall denote by T and Π respectively. Setting the left-hand sides of (22) equal to zero and letting

$$U = \frac{\Pi}{T} \quad , \quad w = \Pi T \quad ; \quad \Pi^2 = wU \quad , \quad T^2 = \frac{w}{U}$$

(23)

$$w = \frac{1}{2}\left\{a_1\cos\theta + a_2\sin\theta \pm \left[(a_1\cos\theta - a_2\sin\theta)^2 + 4\frac{\mu^2\nu^2}{bc}\right]^{1/2}\right\}$$

(24)

$$U = \frac{b}{2\mu\nu}\left\{a_1\cos\theta - a_2\sin\theta \mp \left[(a_1\cos\theta - a_2\sin\theta)^2 + 4\frac{\mu^2\nu^2}{bc}\right]^{1/2}\right\}$$

where

$$a_1 = 2\rho\omega v\beta L\mu \quad , \quad a_2 = 2\rho\omega v\alpha L\mu$$

(25)

Let us assume large magnetic Reynolds numbers so that

$$a_1{}^2, a_2{}^2 \;>> \; \frac{4\mu^2\nu^2}{bc}$$

Then except in the neighborhood of the special value of θ for which $a_1 \cos\theta = a_2 \sin\theta$ we have

$$\left(a_1\cos\theta - a_2\sin\theta\right)^2 \;>>\; \frac{4\mu^2\nu^2}{bc}$$

Expanding (24) in powers of $4\mu^2\nu^2 / \left[bc\left(a_1\cos\theta - a_2\sin\theta\right)\right]$, we obtain

$$\left.\begin{aligned}
w &= a_1\cos\theta - \frac{\mu^2\nu^2}{bc\left(a_2\sin\theta - a_1\cos\theta\right)} + \frac{\mu^4\nu^4}{b^2c^2\left(a_2\sin\theta - a_1\cos\theta\right)^3} + O^{-5}(a_1, a_2) \\[2em]
u &= \frac{\mu\nu}{c\left(a_2\sin\theta - a_1\cos\theta\right)} - \frac{\mu^3\nu^3}{bc^2\left(a_2\sin\theta - a_1\cos\theta\right)^3} + O^{-5}(a_1, a_2)
\end{aligned}\right\} \quad (26)$$

for $a_2\sin\theta > a_1\cos\theta$ and

$$w = a_2\sin\theta - \frac{\mu^2\nu^2}{bc\left(a_1\cos\theta - a_2\sin\theta\right)} + \frac{\mu^4\nu^4}{b^2c^2\left(a_1\cos\theta - a_2\sin\theta\right)^3} + O^{-5}(a_1, a_2)$$

$$(27)$$

$$u = \frac{b}{\mu\nu}\left(a_1\cos\theta - a_2\sin\theta\right) + \frac{\mu\nu}{c\left(a_1\cos\theta - a_2\sin\theta\right)} - \frac{\mu^3\nu^3}{bc^2\left(a_1\cos\theta - a_2\sin\theta\right)^3} + O^{-5}(a_1, a_2)$$

for $a_1\cos\theta > a_2\sin\theta$. The corresponding values for T and Π may be computed:

$a_2\sin\theta > a_1\cos\theta$:

$$\left.\begin{aligned}
T^2 &= \frac{c}{\mu\nu} a_1\cos\theta\left(a_2\sin\theta - a_1\cos\theta\right) + \frac{\mu\nu}{b}\left(\frac{2a_1\cos\theta - a_2\sin\theta}{a_2\sin\theta - a_1\cos\theta}\right) + O^{-2}(a_1, a_2) \\[2em]
\Pi^2 &= \frac{\mu\nu a_1\cos\theta}{c\left(a_2\sin\theta - a_1\cos\theta\right)} - \frac{\mu^3\nu^3 a_1\sin\theta}{bc^2\left(a_2\sin\theta - a_1\cos\theta\right)^3} + O^{-4}(a_1, a_2) \\[2em]
\frac{T^2}{\Pi^2} &= \left(\frac{c}{\mu\nu}\right)^2\left(a_2\sin\theta - a_1\cos\theta\right)^2 + O(1)
\end{aligned}\right\} \quad (28)$$

$a_1\cos\theta > a_2\sin\theta$:

$$T^2 = \frac{\mu\nu\, a_2 \sin\theta}{b(a_1\cos\theta - a_2\sin\theta)} - \frac{\mu^3\nu^3 a_1\cos\theta}{b^2 c\,(a_1\cos\theta - a_2\sin\theta)^3} + O^{-4}(a_1, a_2)$$

$$\Pi^2 = \frac{b}{\mu\nu}\, a_2\sin\theta\,(a_1\cos\theta - a_2\sin\theta) + \frac{\mu\nu}{c}\left(\frac{2a_2\sin\theta - a_1\cos\theta}{a_1\cos\theta - a_2\sin\theta}\right) + O^{-2}(a_1, a_2) \quad (29)$$

$$\frac{T^2}{\Pi^2} = \left(\frac{\mu\nu}{b}\right)^2 \frac{1}{(a_1\cos\theta - a_2\sin\theta)^2} + O^{-4}(a_1, a_2)$$

To demonstrate the stability of these equilibrium states, consider a small perturbation of the system from equilibrium. We let the toroidal field be represented by $T + \tau$ and the poloidal field by $\Pi + \pi$. T and Π satisfy the equilibrium equations, and τ and π are assumed to be small. Substituting into the time dependent equations (22) we obtain

$$\frac{\partial \pi}{\partial t} = A_{11}\pi + A_{12}\tau + O^2(\pi, \tau) \quad , \quad \frac{\partial \tau}{\partial t} = A_{21}\pi + A_{22}\tau + O^2(\pi, \tau) \quad (30)$$

where

$$A_{11} = -\frac{\nu}{L^2} - \frac{bT^2}{\mu L^2} \quad , \quad A_{12} = \frac{b}{\mu L^2}(a_1\cos\theta - 2T\Pi) \bigg\}$$

$$A_{21} = \frac{c}{\mu L^2}(a_1\sin\theta - 2T\Pi) \quad , \quad A_{22} = -\frac{\nu}{L^2} - \frac{c\Pi^2}{\mu L^2} \bigg\} \quad (31)$$

As a solution we let

$$\pi = k_1\exp\omega t \quad , \quad \tau = k_2\exp\omega t \quad (32)$$

Then

$$0 = k_1(A_{11} - \omega) + k_2 A_{12} \quad , \quad 0 = k_1 A_{21} + k_2(A_{22} - \omega) \quad (33)$$

which yields the characteristic equation

$$\omega = \frac{1}{2}\left\{(A_{11} + A_{22}) \pm \left[(A_{11} - A_{22})^2 + 4A_{12}A_{21}\right]^{1/2}\right\} \quad (34)$$

Computing the A_{ij} from (28), (29), and (31) we obtain two roots, viz.

$$\omega_1 = -\frac{2\nu}{L^2} + O^{-2}(a_1, a_2) \quad (35)$$

$$\omega_2 = -\frac{bc}{\mu^2 \nu L^2}\, a_1 \cos\theta \left(a_2 \sin\theta - a_1 \cos\theta\right) + O(1) \tag{36}$$

for $a_2 \sin\theta > a_1 \cos\theta$, and for $a_1 \cos\theta > a_2 \sin\theta$

$$\omega_1 = -\frac{2\nu}{L^2} + O^{-2}(a_1, a_2) \tag{37}$$

$$\omega_2 = -\frac{bc}{\mu^2 \nu L^2}\, a_2 \sin\theta \left(a_1 \cos\theta - a_2 \sin\theta\right) + O(1) \tag{38}$$

giving an exponential return to the equilibrium in all cases.
(35) and (37) represent free decay back to equilibrium; (36) and
(38) represent active degeneration back to equilibrium.

7. The Dynamics of the Solar Dynamo

In the above section we developed in a semi-quantitative manner the dynamics of hydromagnetic dynamos. The calculations were applicable to both stationary and migratory dynamos, though our main interest is in the migratory dynamo at the moment. From (28) we see that the equilibrium values of the toroidal and poloidal fields may be written

$$T^2 = \frac{4\mu L^2 c}{\nu}(\rho\omega v)^2 \alpha\beta \sin\theta \cos\theta + O^0(\rho\omega v) \left.\right\}$$
$$\Pi^2 = O^0(\rho\omega v) \qquad\qquad\qquad (39)$$

near the equator where $a_2 \sin\theta > a_1 \cos\theta$ and from (29)

$$T^1 = O^1(\rho\omega v) \left.\right\}$$
$$\Pi^1 = \frac{4\mu L^1 b}{\nu}(\rho\omega v)^2 \alpha\beta \sin\theta \cos\theta + O^1(\rho\omega v) \qquad (40)$$

near the poles where $a_2 \cos\theta > a_1 \sin\theta$

First, we note that we do not have the purely kinematical result (Bullard, 1954) that there is a critical velocity of the convective motions, which, if exceeded, results in an unlimited exponential growth of the field. The reason is that the dynamo results from Coriolis velocities and not from the primary radial convective velocities: The convective velocities give rise to Coriolis forces, resulting in a dynamo only insofar as the magnetic field does not prevent the manifestation of the Coriolis forces as cyclonic velocities and nonuniform rotation.

Second, supposing that $\rho\omega v$ is large, we see that near

the poles equilibrium occurs for $\Pi \gg T$, whereas near the
equator $T \gg \Pi$. The crossover occurs where $a_1\cos\theta = a_2\sin\theta r$, if
$\alpha = \beta$, at $\theta = 45^\circ$. Thus, above middle solar latitudes we
expect the magnetic activity to arise primarily from poloidal
fields and below middle latitudes from toroidal fields.

Let us organize the above dynamical considerations into a
coherent picture of the solar dynamo. The dynamo was discussed
in section 5 from a purely kinematical point of view: In the upper
half of the solar convective zone we have a relatively weak travel-
ing dynamo wave originating near the poles and being strongly am-
plified as it travels toward the equator. (39) and (40) imply
that it will first be the poloidal field that is the more strongly
amplified, somewhere in the middle latitudes the strong amplifi-
cation of the poloidal field will cease and the toroidal field will
be amplified. We expect then that magnetic phenomena arising from
the poloidal field will be observed primarily toward the pole
from middle latitudes (though of course the poloidal field may be
strong enough to have some observable effects nearer the equator);
phenomena arising from the toroidal field will be observed toward
the equator from the middle latitudes. Few, if any local magnetic
phenomena such as sunspots, flares, and prominences should be
observed at the equator, where both the poloidal and toroidal
fields will be decaying.

Now the observed nonuniform rotation of the sun, wherein the
surface equatorial regions rotate more rapidly than the average of
the sun, (contrary to the effect produced by the Coriolis forces
of the convective motions), can be interpreted in the same manner
as Bullard et al,(1950) has interpreted the corresponding

terrestrial phenomenon, the westerly drift of the Earth's magnetic field. The nonuniform rotation depends on the strength of the toroidal field, and hence theoretically should increase toward the equator from the middle latitudes where it is essentially zero. This is as observed (Kuiper, 1953).

The northern and southern hemispheres of the sun operate as independent dynamos except for whatever loose coupling may exist due to fields diffusing across the equator. The opposite polarity of sunspots in the two hemispheres indicates that the coupling is such that the toroidal field changes in sign across the equator. The looseness of the coupling between hemispheres and between the dynamos of the inner and outer halves of the convective zone explains the lack of complete similarity between hemispheres and between consecutive cycles.

Elsasser[*] has suggested that sunspots arise as a consequence of a strand of the overall mat of toroidal field rising through the photosphere. This gives an inverted "U" shaped loop of flux rising from the main toroidal field up to the surface of the sun, and, together with the migratory dynamo yielding magnetic waves of alternating sign traveling from the poles to the equator, gives a sunspot cycle with the proper reversal of spot polarity between the half cycles of 11 years.

The question now arises as to why we should obtain a migratory dynamo for the sun and a stationary dynamo for the core of the Earth. First we note that below and above the convective zone there is presumably not sufficient turbulence to significantly diffuse magnetic fields. Beneath the convective

---

[*]Unpublished

zone the conductivity is of the order of 10^7 mho/m (Cowling, 1945).
The magnetic viscosity $1/(\mu c)$ becomes 10^{-1} m^2/sec and the time
required to diffuse even 10^5 km is of the order of 10^9 years.
The conductivity above the convective zone is also large, and dif-
fusion upward is ruled out unless, due to some local anomaly, the
field can poke out through the less dense overlying layers pushing
the matter aside, as evidently happens locally in sunspots and
prominences. We conclude that the magnetic fields generated by
convection will be confined largely within the convective shell
itself.

With the great length of the convective zone as compared
to its depth, we do not expect the magnetic activity, whose scale
is determined by the relatively shallow depth of the zone, to be
correlated over the entire solar sphere. Hence, a magnetically
coherent region of the sun, e.g. the northern solar temperate
zone, is isolated and is in an effectively unclosed space. This
is in contrast to the situation in the core of the Earth, where the
depth and circumference of the core are of the same order of
magnitude, resulting in magnetic coherence over the entire closed
space. It is the solution of (4) and (7) in a closed or unclosed
space that yields a stationary or migratory dynamo respectively.

8. Magnetic Stars

Babcock (1953) has shown that many **A** and **M** stars have general magnetic fields of several thousand gauss. Several cases are known to be magnetic variables with periods from several days up to a year. Babcock states that in no case is there good evidence that the magnetic field is without time fluctuations, though many stars exhibit overall magnetic fluctuations which are not periodic. Given a star with a deeper and more active convective zone than the sun there seems no basic difficulty in accounting for the extreme magnetic activity that Babcock observes. From (39) and (40) we see that the field is proportional to $L \rho \omega v / \nu^{1/2}$. If the eddy diffusivity ν is proportional to $L v$ we have $L^{1/2} \rho \omega v^{1/2}$. Increasing L, ω, and v by only a factor of ten from the 10^5 km, 5×10^{-6} sec^{-1}, and 1 m/sec used in the sun gives fields with intensities 10^2 times the solar fields and magnetic periods of two years; increasing v to 1 km/sec gives fields 10^3 times the solar field with a period of one week.

Many magnetic stars exhibit a marked asymmetry in their variations, indicating that they have a large stationary component of their magnetic fields. The discussion at the end of the previous section would imply that their convective zones are proportionately deeper than in the sun, resulting in a large stationary dynamo component along with the migratory dynamo.

Appendix

To consider further solutions of (13) we let

$$b(\underline{k}, t) = p(\underline{k}) \exp \omega t \tag{41}$$

Substituting into (13) we obtain for the equation for $p(\underline{k})$

$$p(\underline{k}) = i \frac{1}{(\omega + \nu k^2)} \int_{-\infty}^{+\infty} dk_1' \int_{-\infty}^{+\infty} dk_2' \frac{(-\underline{e}_x k_2' + \underline{e}_z k_1) \cdot \underline{h}(\underline{k} - \underline{k}')}{(\omega + \nu k'^2)} \tag{42}$$

$$\times \int_{-\infty}^{+\infty} dk_1'' \int_{-\infty}^{+\infty} dk_2'' \, \gamma(\underline{k}' - \underline{k}'') \, p(\underline{k}'')$$

The coupling between the modes $\underline{k}$ and $\underline{k}''$ is given by

$$U(\underline{k}, \underline{k}') = \frac{(-\underline{e}_x k_2 + \underline{e}_z k_1) \cdot \underline{h}(\underline{k} - \underline{k}') \, \gamma(\underline{k}' - \underline{k}'')}{(\omega + \nu k^2)(\omega + \nu k'^2)} \tag{43}$$

$U(\underline{k}, \underline{k}')$ is a monotonically and rapidly decreasing function of both k and k' because of the two factors in the denominator. Usually $\underline{h}(\underline{k} - \underline{k}')$ and $\gamma(\underline{k}' - \underline{k}'')$ are rapidly decreasing functions of the magnitude of their arguments. Thus $U(\underline{k}', \underline{k}'')$ will be small unless $\underline{k}' \sim \underline{k}''$. This means that the coupling to a given mode $\underline{k}$ usually does not extend over a large neighborhood of $\underline{k}$, which allows an approximate solution of (42) without undue calculation.

Consider the solution of (42) when

$$\nabla v = \underline{e}_z H \quad , \quad \Gamma(\underline{r}) = \Gamma E(z) \tag{44}$$

where H and Γ are constants. Their Fourier transforms are

$$\underline{h}(\underline{k}) = \underline{e}_z H \delta(\underline{k}) \quad , \quad g(\underline{k}) = \Gamma \delta(k_1) e(k_2) \tag{45}$$

where

$$e(k_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \, \exp(-ik_2 z) \, \Gamma(z) \tag{46}$$

(42) reduces to the linear homogeneous integral equation

$$p(k_1, k_2) = \lambda(k_1) \int_{-\infty}^{+\infty} dk_2' \, K(k_2, k_2') \, p(k_1, k_2') \qquad (47)$$

where

$$\lambda(k_1) = i \, H \Gamma \, k_1, \qquad K(k_2, k_2') = \frac{e(k_2 - k_2')}{\omega + \nu k^2} \qquad (48)$$

(47) has nontrivial solutions (Lovitt, 1950) if

$$0 = 1 - \lambda(k_1) \int_{-\infty}^{+\infty} dk_2 \, K(k_2, k_2) + \frac{\lambda^2(k_1)}{2!} \int_{-\infty}^{+\infty} dk_2 \int_{-\infty}^{+\infty} dk_2' \begin{vmatrix} K(k_2, k_2) & K(k_2, k_2') \\ K(k_2', k_2) & K(k_2', k_2') \end{vmatrix}$$

$$+ \cdots \qquad (49)$$

(48) is an absolutely and permanently converging series in $\lambda(k_1)$ Thus to a first approximation we drop all terms second order and higher in $\lambda(k_1)$ and obtain the eigenvalue

$$\lambda(k_1) \cong \left[\int_{-\infty}^{+\infty} dk_2 \, K(k_2, k_2) \right]^{-1} \qquad (50)$$

Using (48) and (50) we obtain

$$H \Gamma = - i \, \frac{2 \gamma \nu \, (\omega + \nu k_1^2)^{3/2}}{\pi \, e(o) \, k_1} \qquad (51)$$

Solving for ω we obtain

$$\omega = - \nu k_1^2 + \left[i \, \frac{\pi \, e(o) \, k_1 \, H \Gamma}{2 \gamma \nu} \right]^{2/3} \qquad (52)$$

With $\lambda(k_1)$ given by (50) the solution of (47) is given by the absolutely and uniformly converging series

$$p(\underline{k}) = \lambda(k_1) K(k_2, k_0) - \lambda^2(k_1) \int_{-\infty}^{+\infty} dk \begin{vmatrix} K(k_2, k_0) & K(k_2, k) \\ K(k, k_0) & K(k, k) \end{vmatrix} + \cdots \qquad (53)$$

for suitable k_0. We shall put $k_0 = 0$. Then, to a first approximation

$$p(\underline{k}) \cong \lambda(k_1) K(k_2, 0) = i \, H \Gamma \, \frac{k_1 \, e(k_2)}{(\omega + \nu k^2)^2} \qquad (54)$$

Now $i^{2/3} = -1$ or $\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}$. The first case gives a degenerating traveling dynamo wave and the solution vanishes for large t. Putting $i^{2/3} = \frac{1}{2} \pm i\frac{1}{2}\sqrt{3}$ we obtain the regenerating traveling dynamo wave

$$b(\underline{k},t) = iH\Gamma \frac{k_1 e(k_2)}{\left\{ \left(\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}\right) \left[\frac{\pi e(0) k_1 H\Gamma}{2\nu}\right]^{1/3} + \nu k_2^2 \right\}^2}$$

$$\times \exp\left\{ \left[\frac{1}{2}\left(\frac{\pi e(0) k_1 H\Gamma}{2\sqrt{\nu}}\right)^{2/3} - \nu k^2\right]t \right\} \exp\left[\pm\frac{1}{2}\sqrt{3}\left(\frac{\pi e(0) k_1 H\Gamma}{2\sqrt{\nu}}\right)^{2/3}t\right] \tag{55}$$

traveling with a speed

$$\frac{\sqrt{3}}{2 k_1}\left(\frac{\pi e(0) k_1 H\Gamma}{2\sqrt{\nu}}\right)^{2/3} \tag{56}$$

in the $\underline{k}$ direction.

In a similar manner we may solve (31) in the case that

$$\nabla v = \underline{e}_z H J(z) \quad, \quad \Gamma(\underline{r}) = \Gamma .$$

Their Fourier transforms are

$$\underline{h}(\underline{k}) = \underline{e}_z H \delta(k_1) j(k_2) \quad, \quad \gamma(\underline{k}) = \Gamma \delta(\underline{k})$$

where

$$j(k_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \, \exp(-ik_1 z) J(z) .$$

(42) again reduces to a linear homogeneous integral equation. Defining

$$L(k_2, k_2') = \frac{j(k_2 - k_2')}{(\omega + \nu k^2)(\omega + \nu k'^2)}$$

we obtain

$$P(k_1, k_2) = \lambda(k_1) \int_{-\infty}^{+\infty} dk_2' L(k_2, k_2') P(k_1, k_2') .$$

To a first approximation

$$\lambda(k_1) = \left[\int_{-\infty}^{+\infty} dk_2 \, L(k_1, k_2) \right]^{-1} = \frac{2(\omega + \nu k_1^2)^{3/2}}{\pi j(0)}$$

and

$$\omega = -\nu k_1^2 + \left[i \, \frac{\pi j(0) k_1 H \Gamma}{2\sqrt{\nu}} \right]^{2/3}$$

To a first approximation

$$p(k) = i \, H\Gamma \frac{k_1 \, j(k_2)}{\omega(\omega + \nu k^2)}$$

Neglecting the degenerative solution resulting from $i^{2/3} = -1$, we have the regenerative traveling dynamo wave

$$b(k, t) = i H\Gamma \frac{k_1 \, j(k_2)}{\left(\frac{1}{2} \pm i \frac{1}{2}\sqrt{3} \right) \left[\frac{\pi j(0) k_1 H\Gamma}{2\sqrt{\nu}} \right]^{2/3} \left\{ \left(\frac{1}{2} \pm i \frac{1}{2}\sqrt{3} \right) \left[\frac{\pi j(0) k_1 H\Gamma}{2\nu} \right]^{1/3} + \nu k_2^2 \right\}}$$

$$\times \exp \left\{ \left[\frac{1}{2} \left(\frac{\pi j(0) k_1 H\Gamma}{2\sqrt{\nu}} \right)^{1/3} - \nu k_1^2 \right] t \right\} \exp \left\{ \pm i \frac{1}{2}\sqrt{3} \left(\frac{\pi j(0) k_1 H\Gamma}{2\sqrt{\nu}} \right) t \right\}$$

H. W. Babcock (1953) Mon. Not. Roy. Ast. Soc. <u>113</u> 357

W. M. Elsasser (1946/7) Phys. Rev. <u>69</u> 106, <u>70</u> 202, <u>72</u> 821

E. N. Parker (1954) in print

T. G. Cowling (1945) Mon. Not. Roy. Ast. Soc. <u>105</u> 166

W. V. Lovitt (1950) <u>Linear Integral Equations</u>, Dover (New York)

E. C. Bullard (1954) Phil. Trans. Roy. Soc. in print

E. C. Bullard, C. Freedman, H. Gellman, J. Nixon (1950)
Phil. Trans. A. <u>243</u> 67

G. P. Kuiper Ed. <u>The Sun</u>, Chicago University Press (1953)

Figure 1

Figure 2

(a)

(b)
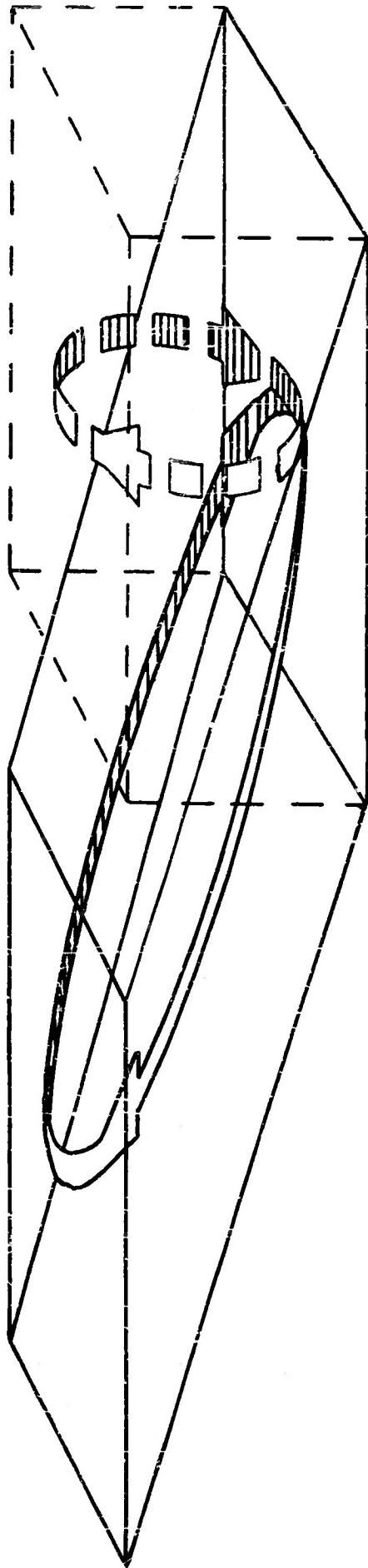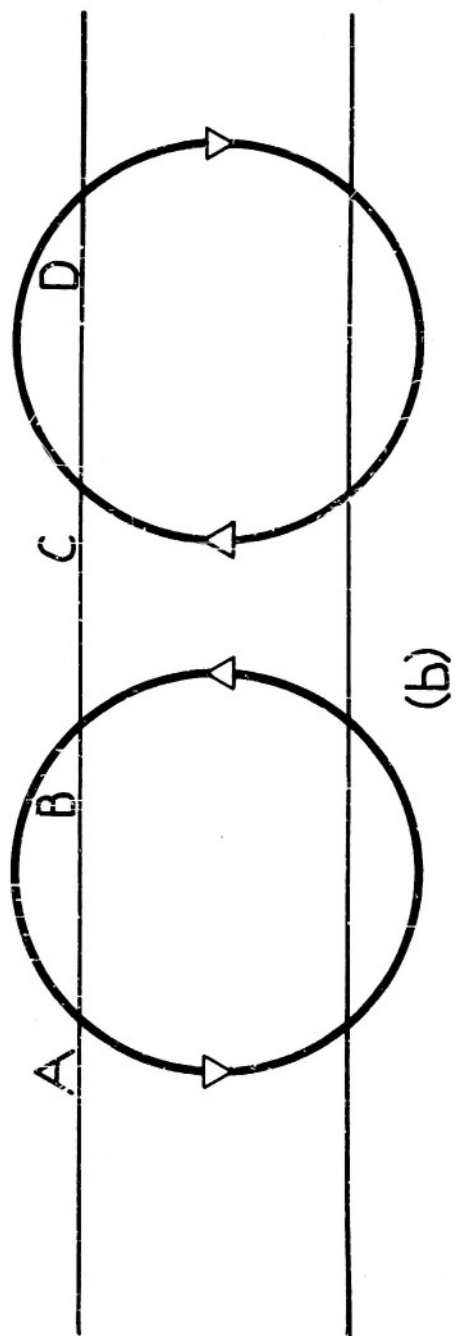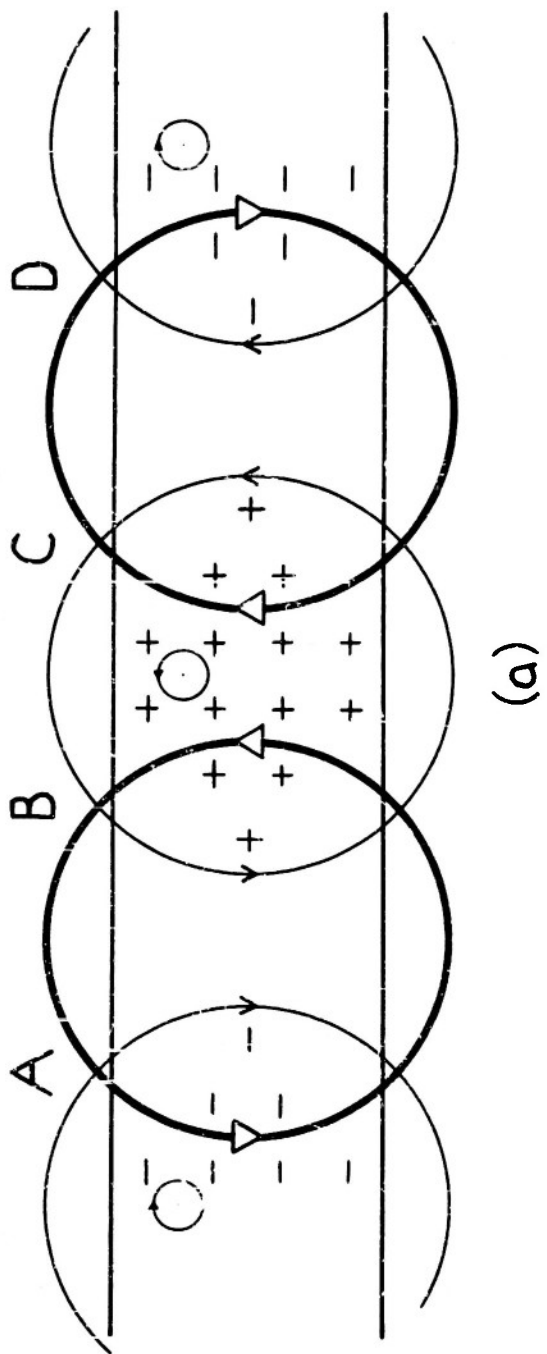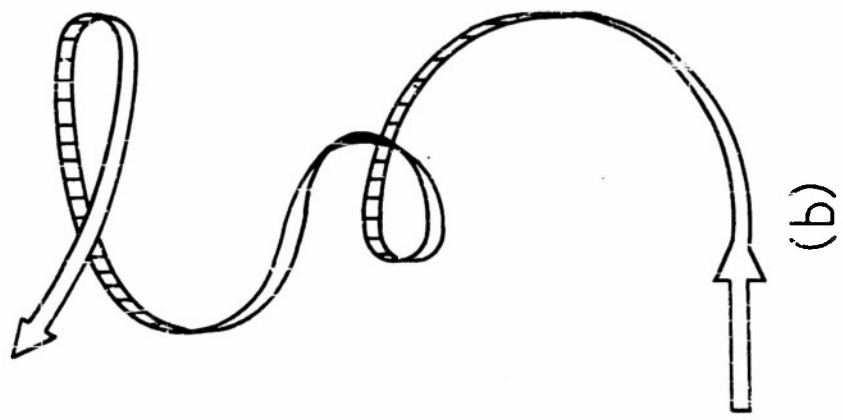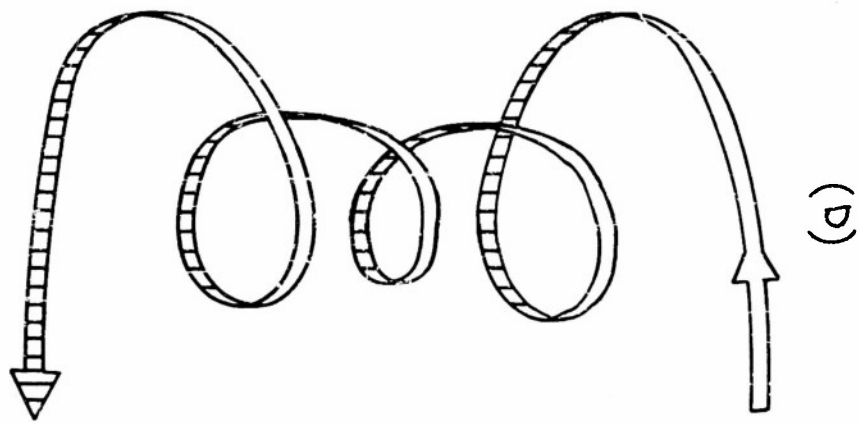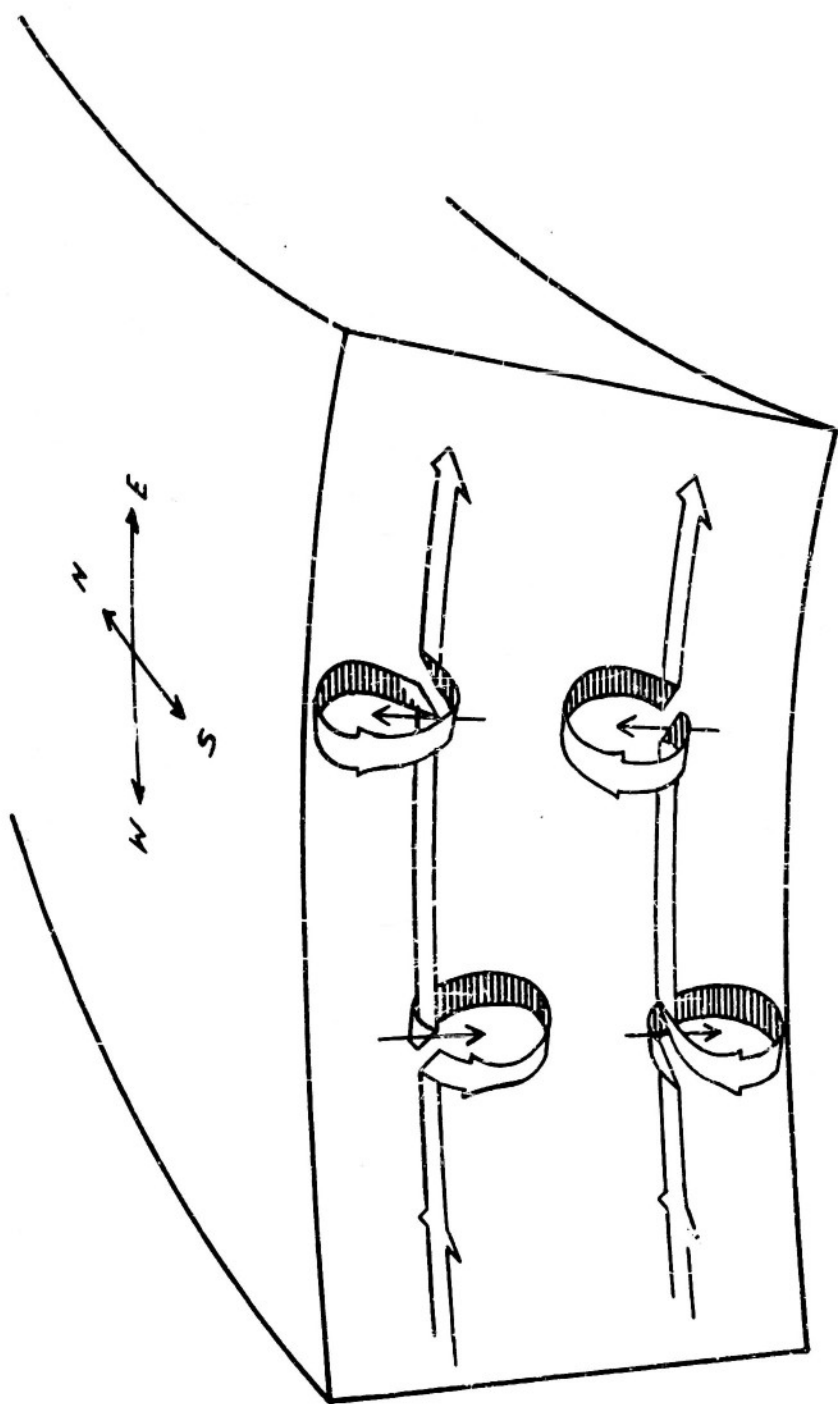
Figure 3

Figure 4

DISTRIBUTION LIST

Addressee	No. of Copies
U.S. Navy Hydrographic Office, Washington 25, D.C.	1
Director, Woods Hole Oceanographic Institution, Woods Hole, Massachusetts	1
Director, Scripps Institution of Oceanography La Jolla, California	1
Director, Lamont Geological Observatory, Torrey Cliff, Palisades, New York	1
Director, U.S. Geological Survey, Department of Interior, Washington 25, D.C.	1
U.S. Geological Survey, Geophysics Branch Washington 25, D.C.	1
Director, Institute of Geophysics, University of California Los Angeles, California	1
Princeton University, Department of Geology, Princeton, New Jersey	1
Carnegie Institution of Washington, Department of Terrestial Magnetism, Washington 25, D.C.	1
Dr. G. P. Wollard, University of Wisconsin, Madison, Wisconsin	1
Geophysics Branch, Code 416, Office of Naval Research Washington 25, D.C.	2
Director, Naval Research Laboratory, Attention: Technical Information Officer, Washington 25, D. C.	6
Officer-in-Charge, Office of Naval Research London Branch Office Navy No. 100, Fleet Post Office, New York, New York	2
Office of Naval Research Branch Office, 346 Broadway New York 13, New York	1
Office of Naval Research Branch Office, Tenth Floor, The John Crerar Library Building, 86 East Randolph Street, Chicago, Illinois	1
Office of Naval Research Branch Office, 1030 East Green Street, Pasadena 1, California	1
Office of Naval Research Branch Office, 1000 Geary Street, San Francisco, California	1
Armed Services Technical Information Center, Documents Service Center, Knott Building, Dayton 2, Ohio	5
Assistant Secretary of Defense for Research and Development, Attention: Committee on Geophysics and Geography, Pentagon Building, Washington 25, D.C.	1
Office of Technical Services, Department of Commerce, Washington 25, D. C.	1

Armed Services Technical Information Agency

AD 38698

Reproduced by

DOCUMENT SERVICE CENTER
KNOTT BUILDING, DAYTON, 2, OHIO

UNCLASSIFIED